International Journal of Theoretical Physics, Vol. 39, No. 5, 2000

Cosmological Relativity: A New Theory of Cosmology[1]

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Received September 9, 1999

A new general-relativistic theory of cosmology, the dynamical variables of which are those of Hubble's, namely distances and redshifts, is presented. The theory describes the universe as having a three-phase evolution with a decelerating expansion followed by a constant and an accelerating expansion, and it predicts that the universe is now in the latter phase. The theory is actually a generalization of Hubble's law taking gravity into account by means of Einstein's theory of general relativity. The equations obtained for the universe expansion are elegant and very simple. It is shown, assuming $\Omega_0 = 0.24$, that the time at which the universe goes over from a decelerating to an accelerating expansion, i.e., the constant expansion phase, occurs at 0.03 τ from the big bang, where τ is the Hubble time in vacuum. Also, at that time the cosmic radiation temperature was 11 K. Recent observations of distant supernovae imply, in defiance of expectations, that the universe's growth is accelerating, contrary to what has always been assumed, that the expansion is slowing down due to gravity. Our theory confirms these recent experimental results by showing that the universe now is definitely in a stage of accelerating expansion.

1. INTRODUCTION

In this paper we present a new theory of cosmology that is based on Einstein's general relativity theory. The theory is formulated in terms of directly measured quantities, i.e., distances, redshifts, and the matter density of the universe.

The general-relativistic theory of cosmology started in 1922 with the remarkable work of Friedmann [1], who solved the Einstein gravitational field equations and found that they admit nonstatic cosmological solutions presenting an expanding universe. Einstein, believing that the universe should

[1] Paper dedicated to Professor Sir Hermann Bondi on the occasion of his 80th birthday.

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0020-7748/00/0500-1375$18.00/0

be static and unchanged forever, suggested a modification to his gravitational field equations by adding to them the so-called cosmological term which can stop the expansion.

Soon after that Hubble [2] found experimentally that the distant galaxies are receding from us, and that the farther the galaxy, the bigger its velocity. In simple words, the universe is indeed expanding according to a simple physical law that gives the relationship between the receding velocity and the distance,

$$\mathbf{v} = H_0\mathbf{R} \tag{1.1}$$

Equation (1.1) is usually referred to as the Hubble law, and H_0 is called the Hubble constant. It is tacitly assumed that the velocity is proportional to the actual measurement of the redshift z of the receding objects by using the nonrelativistic relation $z = \mathbf{v}/c$, where c is the speed of light in vacuum.

The Hubble law does not resemble standard dynamical physical laws that are familiar in physics. Rather, it is a *cosmological equation of state* of the kind one has in thermodynamics such as the one that relates the pressure, volume, and temperature, $pV = RT$ (see e.g., ref. 3). It is this Hubble equation of state that will be extended so as to include gravity by use of the full Einstein theory of general relativity. The obtained results will be very simple, expressing distances in terms of redshifts; depending on the value of $\Omega = \rho/\rho_c$, we will have accelerating, constant, and decelerating expansions corresponding to $\Omega < 1$, $\Omega = 1$, and $\Omega > 1$, respectively. But the last two cases will be shown to be excluded on physical evidence, although the universe had decelerating and constant expansions before it reached its present accelerating expansion stage. *As is well known the standard FRW cosmological theory does not deal directly with Hubble's measured quantities, the distances and redshifts*. Accordingly, the present theory can be compared directly with important recent observations made by astronomers which defy expectations.

In Sections 2 and 3 we review the standard Friedmann and Lemaître models. In Section 4 we mention some weak points in those theories. In Section 5 we present *our cosmological theory written in terms of distances and redshifts*, whereas Section 6 is devoted to the concluding remarks.

2. REVIEW OF THE FRIEDMANN MODELS

Before presenting our theory, and in order to fix the notation, we very briefly review the existing theory [4–7].[3] In the four-dimensional curved space-time describing the universe, our spatial three-dimensional space is assumed to be isotropic and homogeneous. Comoving coordinates, in which

[3] Our notation is similar to that used in ref. 7.

$g_{00} = 1$ and $g_{0k} = 0$, are employed [8, 9]. Here and throughout this paper lowercase Latin indices take the values 1, 2, 3, Greek indices take the values 0, 1, 2, 3, and the signature will be $(+ - - -)$. The four-dimensional spacetime is split into $1 \oplus 3$ parts, and the line element is written as

$$ds^2 = dt^2 - dl^2, \qquad dl^2 = {}^{(3)}g_{kl}\, dx^k\, dx^l = -g_{kl}\, dx^k\, dx^l \tag{2.1}$$

and the 3×3 tensor ${}^{(3)}g_{kl} \equiv -g_{kl}$ describes the geometry of the three-dimensional space at a given instant of time. In the above equations the speed of light c is taken as unity.

Because of the isotropy and homogeneity of the three-geometry, it follows that the curvature tensor must have the form

$${}^{(3)}R_{mnsk} = K[{}^{(3)}g_{ms}{}^{(3)}g_{nk} - {}^{(3)}g_{mk}{}^{(3)}g_{ns}] \tag{2.2}$$

where K is a constant, the curvature of the three-dimensional space, which is related to the Ricci scalar by ${}^{(3)}R = -6K$.[4] By simple geometrical arguments one then finds that

$$dl^2 = (1 - r^2/R^2)^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.3}$$

where $r < R$. Furthermore, the curvature tensor corresponding to the metric (2.3) satisfies Eq. (2.2) with $K = 1/R^2$. In the spherical coordinates (t, r, θ, ϕ) we thus have

$$g_{11} = -(1 - r^2/R^2)^{-1} \tag{2.4}$$

R is called the radius of the curvature (or the expansion parameter) and its value is determined by the Einstein field equations.

One then has three cases: (1) a universe with positive curvature, for which $K = 1/R^2$; (2) a universe with negative curvature, $K = -1/R^2$; and (3) a universe with zero curvature, $K = 0$. The g_{11} component for the negative-curvature universe is given by

$$g_{11} = -(1 + r^2/R^2)^{-1} \tag{2.5}$$

where $r < R$. For the zero-curvature universe one lets $R \to \infty$.

Although general relativity theory asserts that all coordinate systems are equally valid, in this theory one has to change variables in order to get the "right" solutions of the Einstein field equations according to the type of the universe. Accordingly, one makes the substitution $r = R \sin \chi$ for the positive-curvature universe and $r = R \sinh \chi$ for the negative-curvature universe. In addition, the timelike coordinate is also changed into another one η by the transformation $dt = R\, d\eta$. The corresponding line elements then become

[4] For more details on the geometric meaning see ref. 10.

$$ds^2 = R^2(\eta)[d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{2.6a}$$

for the positive-curvature universe,

$$ds^2 = R^2(\eta)[d\eta^2 - d\chi^2 - \sinh^2\chi(d\theta^2 + \sinh^2\theta\, d\phi^2)] \tag{2.6b}$$

for the negative-curvature universe, and

$$ds^2 = R^2(\eta)[d\eta^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{2.6c}$$

for the flat three-dimensional universe. In the sequel, we will see that the timelike coordinate in our theory will take one more different form.

The Einstein field equations are then employed in order to determine the expansion parameter $R(\eta)$. In fact only one field equation is needed,

$$R_0^0 - \tfrac{1}{2}\delta_0^0 R + \Lambda\delta_0^0 = 8\pi G T_0^0 \tag{2.7}$$

where Λ is the cosmological constant, and c is taken as unity. In the Friedmann models one takes $\Lambda = 0$, and in the comoving coordinates used one easily finds that $T_0^0 = \rho$, the mass density. While this choice of the energy-momentum tensor is acceptable in standard general relativity and in Newtonian gravity, we will argue in the sequel that it is not so for cosmology. At any rate, using $\rho(t) = M/2\pi^2R^3$, where M is the "mass" and $2\pi^2R^3$ is the "volume" of the universe, one obtains

$$3[(dR/dt)^2 + 1]/R^2 = 4GM/\pi R^3 + \Lambda \tag{2.8a}$$

or, in terms of η along with taking $\Lambda = 0$,

$$3[(dR/d\eta)^2 + R^2]/R = 4GM/\pi \tag{2.9a}$$

The solution of this equation is

$$R = {}^*R(1 - \cos\eta) \tag{2.10a}$$

where $^*R = 2GM/3\pi$, and from $dt = R\, d\eta$ we obtain

$$t = {}^*R\,(\eta - \sin\eta) \tag{2.11a}$$

Equations (2.10a) and (2.11a) are those of a cycloid, and give a full representation for the expansion parameter of the universe. Figure 1 shows a plot of R as a function of t. At $t = 0, \pm 2\pi{}^*R, \pm 4\pi{}^*R, \ldots$, etc., $R(t)$ vanishes; that is, the universe contracts to a point. Since the density will become very large when this is about to happen, our approximate expression for the energy-momentum tensor will fail. We should also keep in mind that the classical Einstein equation becomes inapplicable at very high densities. It is therefore not clear exactly what happens at the singular points of Fig. 1, and we do not know whether the universe actually has the periodic behavior suggested by this figure.

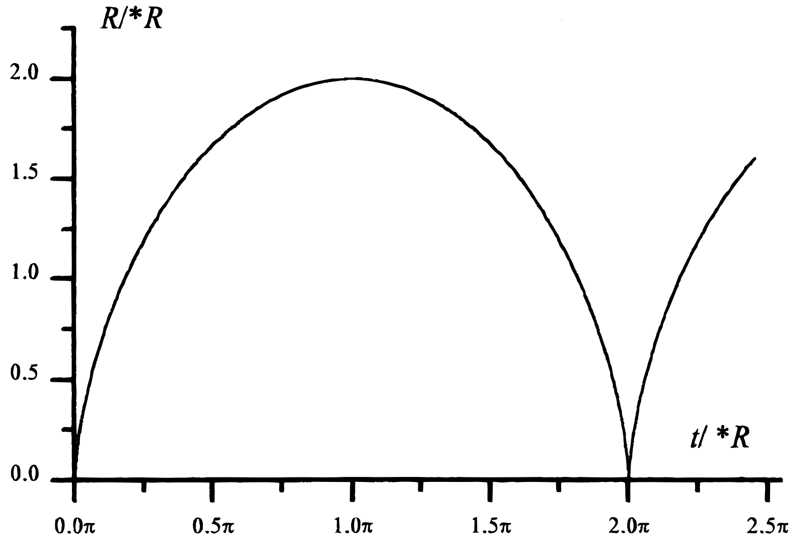

Fig. 1. Radius of curvature of the positive-curvature Friedmann universe as a function of time. The curve is a cycloid.

Similarly, one obtains for the negative-curvature universe the analogs to Eqs. (2.8a) and (2.9a),

$$3[(dR/dt)^2 - 1]/R^2 = 4GM/\pi R^3 + \Lambda \tag{2.8b}$$

$$3[(dR/d\eta)^2 - R^2]/R = 4GM/\pi \tag{2.9b}$$

the solution of which is given by

$$R = {}^*R(\cosh \eta - \eta) \tag{2.10b}$$

$$t = {}^*R(\sinh \eta - \eta) \tag{2.11b}$$

Figure 2 shows R as a function of t. The universe begins with a big bang and continues to expand forever. As $t \to \infty$, the universe gradually becomes flat. Again, the state near the singularity at $t = 0$ is not adequately described by our equations.

Finally, for the universe with a flat three-dimensional space the Einstein field equations yield the analogs to Eqs. (2.8a) and (2.9a),

$$3(dR/dt)^2/R^2 = 4GM/\pi R^3 + \Lambda \tag{2.8c}$$

$$3(dR/d\eta)^8/R = 4GM/\pi \tag{2.9c}$$

As a function of t, the solution is

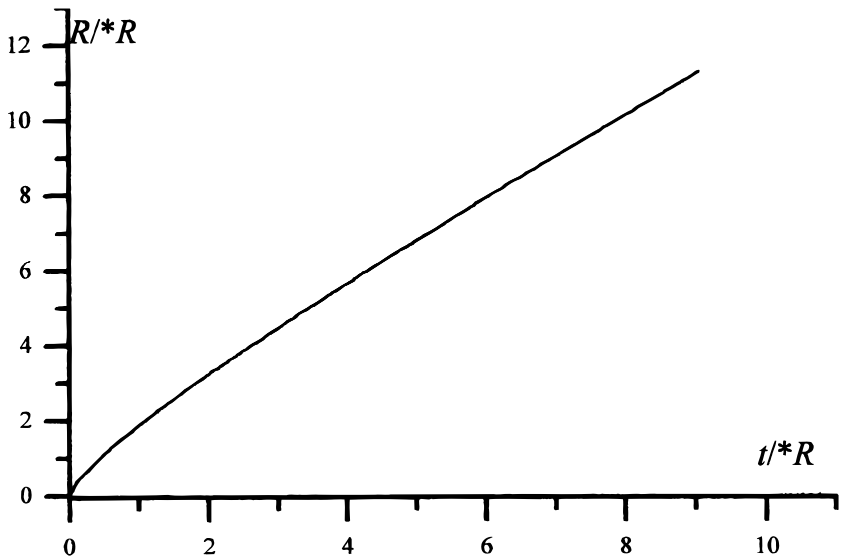

Fig. 2. Radius of curvature of the negative-curvature Friedmann universe as a function of time.

$$R = (3GM/\pi)^{1/3}t^{2/3} \qquad (2.10c)$$

This function is plotted in Fig. 3. As $t \to \infty$, the four-geometry tends to become flat.

3. LEMAÎTRE MODELS

An extension of the Friedmann models was carried out by Lemaître, who considered universes with zero energy-momentum but with a nonzero cosmological constant. While these models are of interest mathematically, they have little, if any, relation to the physical universe because we know that there is baryonic matter. The behavior of the universe in this model will be determined by the cosmological term; this term has the same effect as a uniform mass density $\rho_{eff} = -\Lambda/4\pi G$, which is constant in space and time. A positive value of Λ corresponds to a negative effective mass density (repulsion), and a negative value of Λ corresponds to a positive mass density (attraction). Hence, we expect that in a universe with a positive value of Λ, the expansion will tend to accelerate; whereas in a universe with negative value of Λ, the expansion will slow down, stop, and reverse.

The equations of motion for $R(t)$ have been derived in Section 2, but here it will be assumed that $\Lambda \neq 0$, whereas the energy-momentum tensor appearing in Eq. (2.7) is zero. For the positive-curvature universe one obtains the analog to Eq. (2.8a),

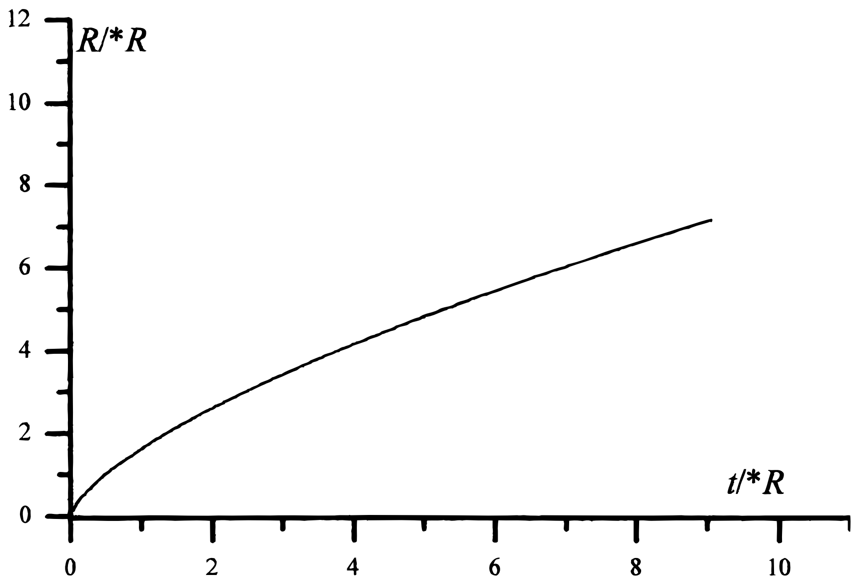

Fig. 3. Radius of curvature of the flat Friedmann universe as a function of time.

$$3[(dR/dt)^2 + 1]/R^2 = \Lambda \tag{3.1a}$$

From Eq. (3.1a) one immediately concludes that $-1 + \Lambda R^2/3$ cannot be negative. This implies that $\Lambda > 0$, and that the value of R can never be less than $(3/\Lambda)^{1/2}$, i.e., the radius of curvature cannot be zero, which excludes the possibility of a big bang.

The integration of Eq. (3.1a) yields

$$R(t) = (3/\Lambda)^{1/2} \cosh[(\Lambda/3)^{1/2}t] \tag{3.2a}$$

where t was taken zero when R has its minimum value. Figure 4 curve (a), shows a plot of R as a function of t. As is seen, for $t > 0$ the universe expands monotonically, and as t increases, R increases, too, and the universe becomes flat.

Similarly, one obtains for the negative-curvature universe the analog to Eq. (3.1a),

$$3[(dR/dt)^2 - 1]/R^2 = \Lambda \tag{3.1b}$$

the integration of which gives

$$R(t) = (3/\Lambda)^{1/2} \sinh[(\Lambda/3)^{1/2}t] \tag{3.2b}$$

for $\Lambda > 0$, and

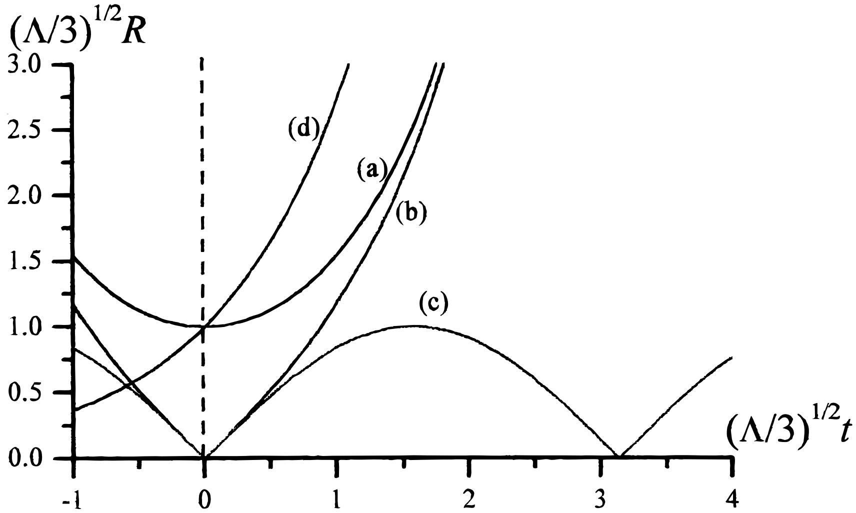

Fig. 4. Radius of curvature of the empty Lemaitre universes as a function of time. (a) Positive-curvature model, $\Lambda > 0$. (b) Negative-curvature model, $\Lambda > 0$. (c) Negative-curvature model, $\Lambda < 0$. (d) Flat model, $\Lambda > 0$.

$$R(t) = (3/-\Lambda)^{1/2} \sinh [(-\Lambda/3)^{1/2} t] \tag{3.2c}$$

for $\Lambda < 0$. These functions are plotted in Fig. 4, curves (b) and (c), respectively. Note that both universes begin with a big bang at $t = 0$. The first of these curves expands monotonically, whereas the second one oscillates. In our actual universe, the mass density near the singularity at $t = 0$ was extremely large, and hence this model cannot be used to describe its behavior near this time.

Finally, for the universe with a flat three-dimensional space the Einstein field equations yield the analog to Eq. (3.1a),

$$3(dR/dt)^2/R^2 = \Lambda \tag{3.1c}$$

This equation has meaning only for $\Lambda > 0$, and it has the solution

$$R(t) = R(0) \exp[(\Lambda/3)^{1/2} t] \tag{3.2d}$$

This universe expands exponentially. This model, described by Eq. (3.2d), is usually called the de Sitter universe.

4. REMARKS AND CRITIQUE OF THE STANDARD THEORY

To conclude the discussion on the Friedmann and Lemaître universes, we briefly discuss the case in which both the matter density and the cosmological constant are not zero. Note that exact solutions of the differential equations describing the expansion of the universe in that case, given by Eqs. (2.8), are not known. These general models can be thought of as a combination of the Friedmann and Lemaître models.

Consider a universe that begins with a big bang. At an early time, the universe must have been very dense, and we can neglect the cosmological term. Hence, we have approximately a Friedmann universe. As the universe expands and the mass density decreases, the cosmological term will become more important. In the Friedmann models of zero and negative curvature, the universe expands monotonically and the decrease in mass density is monotonic, too. The cosmological term will ultimately dominate the behavior of the universe, and it gradually turns into an empty Lemâtre universe with zero or negative curvature. In the case of negative curvature with $\Lambda < 0$, the expansion will stop at some later time, reverse, and finally end up in a recontracting universe of negative curvature.

In the case of a Friedmann universe with a positive curvature, the mass density reaches a minimum when the radius of curvature is at its maximum. Hence, the cosmological term will dominate the behavior of the universe only if it is sufficiently large compared with the minimum mass density. The critical value of Λ is given by $\Lambda_E = (\pi/2GM)^2$. If Λ is larger than Λ_E, then the Friedmann universe with positive curvature gradually turns into an expanding Lemaître universe with positive curvature (Fig. 5). In the case

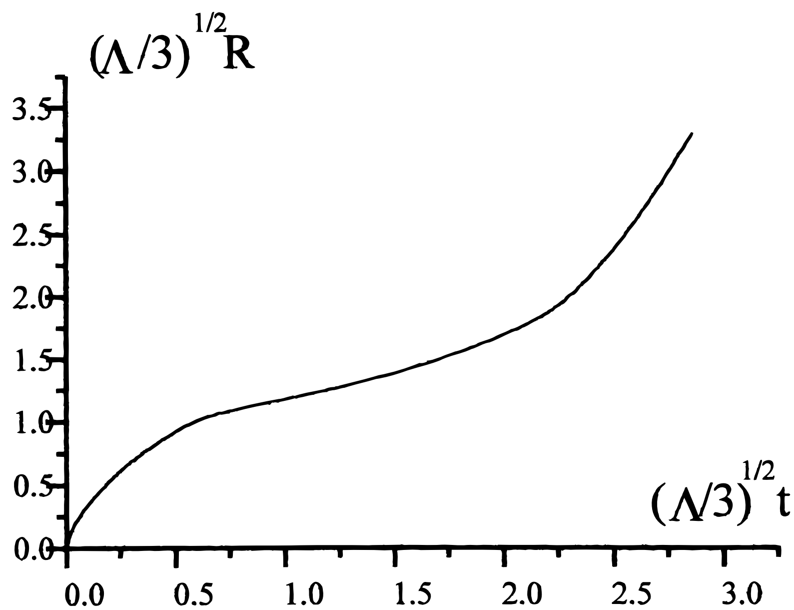

Fig. 5. Radius of curvature of a nonempty Lemaître universe, with $\Lambda > \Lambda_E$.

$\Lambda = \Lambda_E$, the transition is never completed and the expansion stops at a constant value of $R = 1/\Lambda_E^{1/2}$. This static universe is called the Einstein universe. The universe at this value of R, however, is not stable. Any perturbation in R leads either to monotonic expansion (toward an empty Lemaître model) or to contraction (toward a contracting Friedmann universe).

In the final analysis, it follows that the expansion of the universe is determined by the so-called cosmological parameters. These can be taken as the mass density ρ, the Hubble constant H, and the deceleration parameter q. In the following we give a brief review of these parameters and the relationship between them. In the rest of the paper we will concentrate on the theory with dynamical variables that are actually measured by astronomers: distances, redshifts, and the mass density.

Equations (2.8) can be written as

$$3(H^2 + k/R^2) = 8\pi G\rho + \Lambda \tag{4.1}$$

where $k = 1$, $k = 0$, or $k = -1$, for the cases of positive, zero, or negative curvature, respectively. Using Eq. (4.1) and $\Omega = \rho/\rho_c$, where $\rho_c = 3H_0^2/8\pi G$, one obtains

$$\Omega = 1 + k/H^2R^2 - \Lambda/3H^2 \tag{4.2}$$

It follows from these equations that the curvature of the universe is determined by H, ρ, and Λ, or equivalently, H, Ω, and Λ:

$$\Omega > 1 - \Lambda/3H^2 \tag{4.3a}$$

for positive curvature,

$$\Omega < 1 - \Lambda/3H^2 \tag{4.3b}$$

for negative curvature, and

$$\Omega = 1 - \Lambda/3H^2 \tag{4.3c}$$

for zero curvature.

The deceleration parameter is defined as

$$q \equiv -[1 + (1/H^2)\, dH/dt] \tag{4.4}$$

and it can be shown that

$$q = \Omega/2 - \Lambda/3H^2 \tag{4.5}$$

Using Eq. (4.5), we can eliminate Λ from Eqs. (4.3) and obtain

$$3\Omega/2 > 1 + q \tag{4.6a}$$

for positive curvature,

$$3\Omega/2 < 1 + q \tag{4.6b}$$

for negative curvature, and

$$3\Omega/2 = 1 + q \tag{4.6c}$$

for zero curvature.

It is worthwhile mentioning some weak points in the Friedmann theory. One of the assumptions is that the type of the universe is determined by $\Omega = \rho/\rho_c$, where $\rho_c = 3H_0^2/8\pi G$, which requires that the sign of $(\Omega - 1)$ must not change throughout the evolution of the universe so as to change the kind of the universe from one to another. That means that in this theory the universe has only one kind of curvature throughout its evolution and cannot go from one curvature to another. It is not obvious, however, that this is indeed a valid assumption theoretically or experimentally. In other words, the universe has been and will be in only one form of expansion. As will be shown in the sequel, the universe has actually three phases of expansion, and it *does* go from one to the second and then to the third phase.

In the combined Friedmann–Lemaître theory discussed above in which both the matter density and the cosmological constant are not zero, nevertheless, the theory does permit the change of sign of the decelerating parameter q, as can be seen from Fig. 5. There exist no equations, however, that describe this kind of transfer from one type of universe to another.

Finally, one can also argue that astronomers do not measure the radius of curvature of the universe. In fact, one may ask what is the meaning of such a notion for the open universe?

5. COSMOLOGICAL THEORY IN TERMS OF DISTANCE AND REDSHIFT

A new outlook on the universe' expansion can be achieved and is presented here. The new theory has the following features: (1) It gives a direct relationship between distances and redshifts. (2) It is fully general relativistic. (3) It includes two universal constants, the speed of light in vacuum c and the Hubble time in the absence of gravity τ (might also be called the *Hubble time in vacuum*). (4) The redshift parameter z is taken as the timelike coordinate. (5) The energy-momentum tensor is represented differently by including in it a term which is equivalent to the cosmological constant. And (6) it predicts that the universe has three phases of expansion: decelerating, constant, and accelerating, but it is now in the stage of accelerating expansion phase after having gone through the other two phases.

Our starting point is Hubble's cosmological equation of state, Eq. (1.1). One can keep the velocity $\mathbf{v}$ in Eq. (1.1) or replace it with the redshift parameter z by means of $z = \mathbf{v}/c$. Since $\mathbf{R} = (x_1, x_2, x_3)$, the square of Eq. (1.1) then yields

$$c^2 H_0^{-2} z^2 - (x_1^2 + x_2^2 + x_3^2) = 0 \tag{5.1}$$

Our aim is to write our equations in an invariant way so as to enable us to extend them to curved space. Equation (5.1) is not invariant since H_0^{-1} is the Hubble time at present. At the limit of zero gravity, Eq. (5.1) will have the form

$$c^2 \tau^2 z^2 - (x_1^2 + x_2^2 + x_3^2) = 0 \tag{5.2}$$

where τ is Hubble's time in vacuum, which is a *universal constant*, the numerical value of which will be determined in the sequel by relating it to H_0^{-1} at different situations. Equation (5.2) provides the basis of a cosmological special relativity and has been investigated extensively [11–16].

In order to make Eq. (5.2) adaptable to curved space we write it in a differential form:

$$c^2 \tau^2\, dz^2 - (dx_1^2 + dx_2^2 + dx_3^2) = 0 \tag{5.3}$$

or, in a covariant form in flat space,

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = 0 \tag{5.4a}$$

where $\eta_{\mu\nu}$ is the ordinary Minkowskian metric, and our coordinates are $(x^0, x^1, x^2, x^3) = (c\tau z, x_1, x_2, x_3)$. Equation (5.4a) expresses the null condition, familiar from light propagation in space, but here it expresses the universe expansion in space. The generalization of Eq. (5.4a) to a covariant form in curved space can immediately be made by replacing the Minkowskian metric $\eta_{\mu\nu}$ by the curved Riemannian geometrical metric $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = 0 \tag{5.4b}$$

obtained from solving the Einstein field equations.

Because of the spherical symmetry nature of the universe, the metric we seek is of the form [8]

$$ds^2 = c^2 \tau^2\, dz^2 - e^\lambda\, dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \tag{5.5}$$

where comoving coordinates, as in the Friedmann theory, are used and λ is a function of the radial distance r. The metric (5.5) is static and solves the Einstein field equation (2.7). When looking for static solutions, Eq. (2.7) can also be written as

$$e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 = 8\pi G T_0^0 \tag{5.6}$$

when Λ is taken zero, and where a prime denotes differentiation with respect to r.

In general relativity theory one takes $T_0^0 = \rho$. In Newtonian gravity one has the Poisson equation $\nabla^2 \phi = 4\pi G\rho$. At points where $\rho = 0$ one solves the vacuum Einstein field equations and the Laplace equation $\nabla^2\phi = 0$ in Newtonian gravity. In both theories a null (zero) solution is allowed as a trivial case. In cosmology, however, there exists no situation at which ρ can be zero because the universe is filled with matter. In order to be able to have zero on the right-hand side of Eq. (5.6) we take T_0^0 not as equal to ρ, but to $\rho - \rho_c$, where ρ_c is chosen by us now as a *constant* given by $\rho_c = 3/8\pi G\tau^2$.

The introduction of ρ_c in the energy-momentum tensor might be regarded as adding a cosmological constant to the Einstein field equations. But this is not exactly so, since the addition of $-\rho_c$ to T_0^0 means also fixing the numerical value of the cosmological constant and is no longer a variable to be determined by experiment. At any rate our reasons are philosophically different from the standard point of view, and this approach has been presented and used in earlier work [17].

The solution of Eq. (5.6) with $T_0^0 = \rho - \rho_c$ is given by

$$e^{-\lambda} = 1 - (\Omega - 1)r^2/c^2\tau^2 \tag{5.7}$$

where $\Omega = \rho/\rho_c$. Accordingly, if $\Omega > 1$, we have $g_{rr} = -(1 - r^2/R^2)^{-1}$, where

$$R^2 = c^2\tau^2/(\Omega - 1) \tag{5.8a}$$

exactly equals g_{11} given by Eq. (2.4) for the positive-curvature Friedmann universe that is obtained in the standard theory by purely geometrical manipulations (see Section 2). If $\Omega < 1$, we can write $g_{rr} = -(1 + r^2/R^2)^{-1}$ with

$$R^2 = c^2\tau^2/(1 - \Omega) \tag{5.8b}$$

which is equal to g_{11} given by Eq. (2.5) for the negative-curvature Friedmann universe. In the above equations $r < R$.

In Fig. 6 a plot of R as a function of Ω, according to Eqs. (5.8), is given. One can interpret R as the boundary of the universe within which matter can exist, although it is not necessarily that the matter fills up all the space bounded by R.

Moreover, we know that the Einstein field equations for these cases are given by Eqs. (2.8), which, in our new notation, have the form

$$[(dR/dz)^2 + c^2\tau^2]/R^2 = (\Omega - 1) \tag{5.9a}$$

$$[(dR/dz)^2 - c^2\tau^2]/R^2 = (\Omega - 1) \tag{5.9b}$$

As is seen from these equations, if one neglects the first term in the square brackets with respect to the second ones, R^2 will be exactly reduced to their values given by Eqs. (5.9).

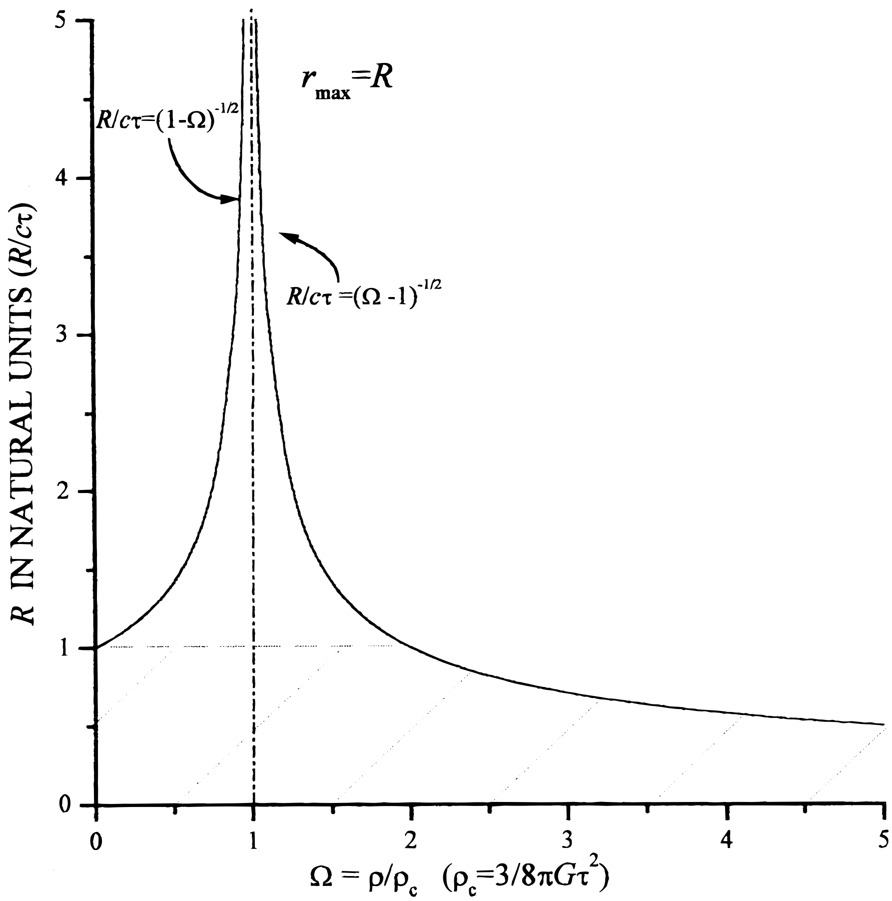

Fig. 6. A plot of R as a function of Ω according to Eqs. (5.8). R is the boundary of the universe within which matter can exist, although it is not necessarily that the matter fills up all the space bounded by R.

The expansion of the universe can now be determined from the null condition $ds = 0$, Eq. (5.4b), using the metric (5.5). Since the expansion is radial, one has $d\theta = d\phi = 0$, and the equation obtained is

$$dr/dz = c\tau[1 + (1 - \Omega)r^2/c^2\tau^2]^{1/2} \tag{5.10}$$

The second term in the square bracket of Eq. (5.10) represents the deviation from constant expansion due to gravity. For without this term, Eq. (5.10) reduces to $dr/dz = c\tau$ or $dr/dv = \tau$, thus $r = \tau v +$ const. The constant can be taken zero if one assumes, as usual, that at $r = 0$ the velocity should also vanish. Accordingly we have $r = \tau \mathbf{v}$ or $\mathbf{v} = \tau^{-1} r$. When $\Omega = 1$, namely when $\rho = \rho_c$, we have a constant expansion.

The equation of motion (5.10) can be integrated exactly by the substitutions

$$\sin \chi = \alpha r / c\tau; \qquad \Omega > 1 \tag{5.11a}$$

$$\sinh \chi = \beta r / c\tau; \qquad \Omega < 1 \tag{5.11b}$$

where

$$\alpha = (\Omega - 1)^{1/2}, \qquad \beta = (1 - \Omega)^{1/2} \tag{5.12}$$

For the $\Omega > 1$ case a straightforward calculation using Eq. (5.11a) gives

$$dr = (c\tau/\alpha) \cos \chi \, d\chi \tag{5.13}$$

and the equation of the universe expansion (5.10) yields

$$d\chi = \alpha \, dz \tag{5.14a}$$

The integration of this equation gives

$$\chi = \alpha z + \text{const} \tag{5.15a}$$

The constant can be determined using Eq. (5.11a). At $\chi = 0$, we have $r = 0$ and $z = 0$, thus

$$\chi = \alpha z \tag{5.16a}$$

or, in terms of the distance, using (5.11a) again,

$$r(z) = (c\tau/\alpha) \sin \alpha z; \qquad \alpha = (\Omega - 1)^{1/2} \tag{5.17a}$$

This is obviously a decelerating expansion.

For $\Omega < 1$, using Eq. (5.11b), a similar calculation yields for the universe expansion (5.10)

$$d\chi = \beta \, dz \tag{5.14b}$$

thus

$$\chi = \beta z + \text{const} \tag{5.15b}$$

Using the same initial conditions as above then gives

$$\chi = \beta z \tag{5.16b}$$

and in terms of distances,

$$r(z) = (c\tau/\beta) \sinh \beta z; \qquad \beta = (1 - \Omega)^{1/2} \tag{5.17b}$$

This is now an accelerating expansion.

For $\Omega = 1$ we have, from Eq. (5.10),

$$d^2r/dz^2 = 0 \tag{5.14c}$$

The solution is, of course,

$$r(z) = c\tau z \tag{5.17c}$$

This is a constant expansion.

It will be noted that the last solution can also be obtained directly from the previous two for $\Omega > 1$ and $\Omega < 1$ by going to the limit $z \to 0$, using L'Hôspital's lemma, showing that our solutions are consistent. It will be shown later that the constant expansion is just a transition stage between the decelerating and the accelerating expansions as the universe evolves toward its present situation.

Figure 7 describes the Hubble diagram of the above solutions for the three types of expansion for values of Ω from 100 to 0.24. The figure describes

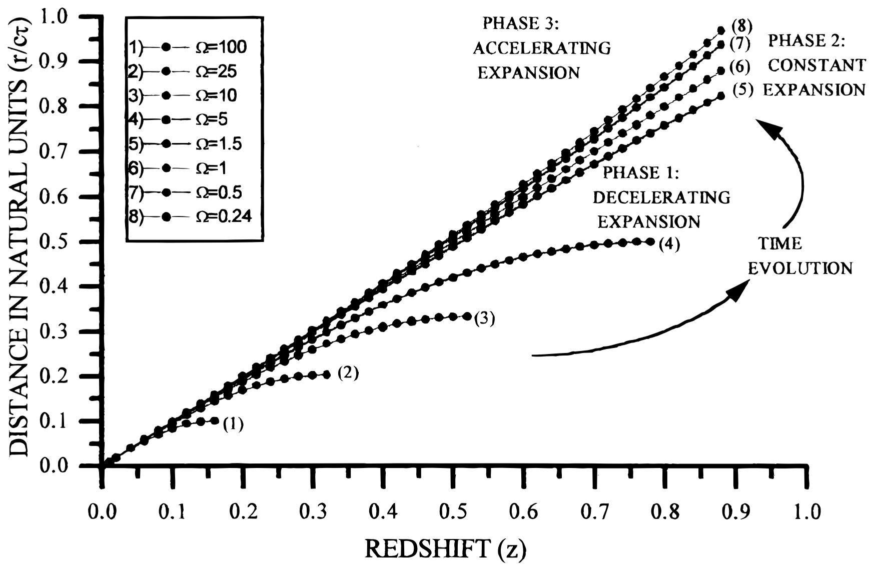

Fig. 7. Hubble's diagram describing the three-phase evolution of the universe according to Einstein's general relativity theory. Curves (1)–(5) represent the stages of *decelerating* expansion according to $r(z) = (c\tau/\alpha) \sin \alpha z$, where $\alpha = (\Omega - 1)^{1/2}$, $\Omega = \rho/\rho_c$, with ρ_c a *constant*, $\rho_c = 3/8\pi G\tau^2$, and c and τ are the speed of light and the Hubble time in vacuum (both universal constants). As the density of matter ρ decreases, the universe goes over from the lower curves to the upper ones, but it does not have enough time to close up to a big crunch. The universe subsequently goes to curve (6) with $\Omega = 1$, at which time it has a *constant* expansion for a fraction of a second. This then followed by going to the upper curves (7)–(8) with $\Omega < 1$, where the universe expands with acceleration according to $r(z) = (c\tau/\beta) \sinh \beta z$, where $\beta = (1 - \Omega)^{1/2}$. One of these last curves fits the present situation of the universe.

Table I. The Cosmic Times with Respect to the Big Bang and the Cosmic Temperature for Each of the Curves in Fig. 7[a]

Curve no.	Ω	Time in units of τ	Time (sec)	Temperature (K)
1	100	3.1×10^{-6}	1.1×10^{12}	1114.0
2	25	9.8×10^{-5}	3.6×10^{13}	279.0
3	10	3.0×10^{-4}	1.1×10^{14}	111.0
4	5	1.2×10^{-3}	4.4×10^{14}	56.0
5	1.5	1.3×10^{-2}	4.7×10^{15}	17.0
6	1	3.0×10^{-2}	1.1×10^{16}	11.0
7	0.5	1.3×10^{-1}	4.7×10^{16}	6.0
8	0.245	1.0	3.6×10^{17}	2.7

[a] The calculations are made using Carmeli's Lorentz-like transformation that relates physical quantities at different cosmic times when gravity is extremely weak [13].

the three-phase evolution of the universe. Curves (1)–(5) represent the stages of decelerating expansion according to Eq. (5.17a). As the density of matter ρ decreases, the universe goes over from the lower curves to the upper ones, but it does not have enough time to close up to a big crunch. The universe subsequently goes to curve (6) with $\Omega = 1$, at which time it has a constant expansion for a fraction of a second. This then followed by going to the upper curves (7) and (8) with $\Omega < 1$, where the universe expands with acceleration according to Eq. (5.17b). A curve of this kind fits the present situation of the universe. For curves (1)–(4) in the diagram we use the cutoff when the curves were at their maximum (or the same could be done by using the cutoff as determined by R of Fig. 6). In Table I we present the cosmic times with respect to the big bang and the cosmic radiation temperature for each of the curves in Fig. 7.

In order to decide which of the three cases is the appropriate one at the present time, we have to write the solutions (5.17) in the ordinary Hubble law form $\mathbf{v} = H_0 r$. To this end we change variables from the redshift parameter z to the velocity $\mathbf{v}$ by means of $z = \mathbf{v}/c$ for $\mathbf{v}$ much smaller than c. For higher velocities this relation is not accurate and one has to use a Lorentz transformation in order to relate z to $\mathbf{v}$. A simple calculation then shows that, for receding objects, one has the relations

$$z = [(1 + \mathbf{v}/c)/\,(1 - \mathbf{v}/c)]^{1/2} - 1 \tag{5.18a}$$

$$\mathbf{v}/c = z(z + 2)/(z^2 + 2z + 2) \tag{5.18b}$$

We will assume that $\mathbf{v} << c$ and consequently Eqs. (5.17) have the forms

$$r(\mathbf{v}) = (c\tau/\alpha)\sin(\alpha\mathbf{v}/c) \tag{5.19a}$$

$$r(\mathbf{v}) = (c\tau/\beta)\sinh(\beta\mathbf{v}/c) \tag{5.19b}$$

$$r(\mathbf{v}) = \tau\mathbf{v} \tag{5.19c}$$

Expanding now Eqs. (5.19a) and (5.19b) and keeping the appropriate terms then yields

$$r = \tau\mathbf{v}(1 - \alpha^2\mathbf{v}^2/6c^2) \tag{5.20a}$$

for the $\Omega > 1$ case, and

$$r = \tau\mathbf{v}(1 + \beta^2\mathbf{v}^2/6c^2) \tag{5.20b}$$

for $\Omega < 1$. Using now the expressions for α and β given by Eq. (5.12) in Eqs. (5.20), both of the latter can be reduced into a single equation

$$r = \tau\mathbf{v}[1 + (1 - \Omega)\mathbf{v}^2/6c^2] \tag{5.21}$$

Inverting now this equation by writing it in the form $\mathbf{v} = H_0 r$, we obtain in the lowest approximation for H_0 the following:

$$H_0 = h[1 - (1 - \Omega)\mathbf{v}^2/6c^2] \tag{5.22}$$

where $h = \tau^{-1}$. Using $\mathbf{v} \approx r/\tau$, or $z = \mathbf{v}/c$, we also obtain

$$H_0 = h[1 - (1 - \Omega)r^2/6c^2\tau^2] = h[1 - (1 - \Omega)z^2/6] \tag{5.23}$$

Consequently, H_0 depends on the distance, or equivalently, on the redshift. As is seen, H_0 has meaning only for $r \to 0$ or $z \to 0$, namely when measured *locally*, in which case it becomes h.

6. CONCLUDING REMARKS

In recent years observers have argued for values of H_0 as low as 50 and as high as 90 km/sec-Mpc; some of the recent ones show 80 $\pm$ 17 km/sec-Mpc [18–26]. There are the so-called "short" and "long" distance scales, with the higher and the lower values for H_0, respectively [27]. Indications are that the longer the distance of measurement, the smaller the value of H_0. By Eqs. (5.22) and (5.23) this is possible only for the case in which $\Omega < 1$, namely when the universe is at an accelerating expansion. Figures 8 and 9 show the Hubble diagrams for the distance–redshift relationship predicted by theory for the accelerating expanding universe at the present time, and Fig. 10 and 11 show the experimental results [28,29].

Our estimate for h, based on published data, is $h \approx$ 85–90 km/sec-Mpc. Assuming $\tau^{-1} \approx 85$ km/sec-Mpc, Eq. (5.23) then gives

$$H_0 = h[1 - 1.3 \times 10^{-4}(1 - \Omega)r^2] \tag{6.1}$$

where r is in Mpc. A computer best-fit can then fix both h and Ω.

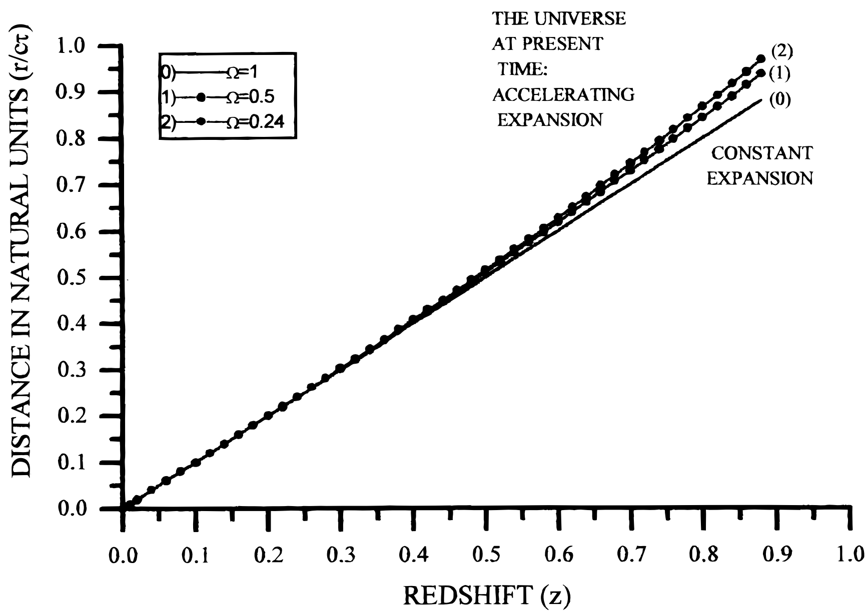

Fig. 8. Hubble's diagram of the universe at the present phase of evolution with accelerating expansion.

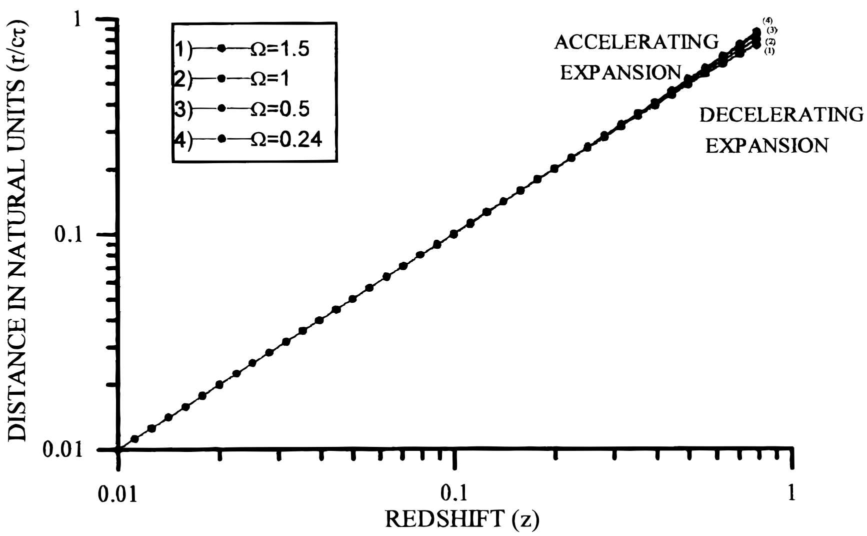

Fig. 9. Hubble's diagram describing decelerating, constant, and accelerating expansions in a logarithmic scale.

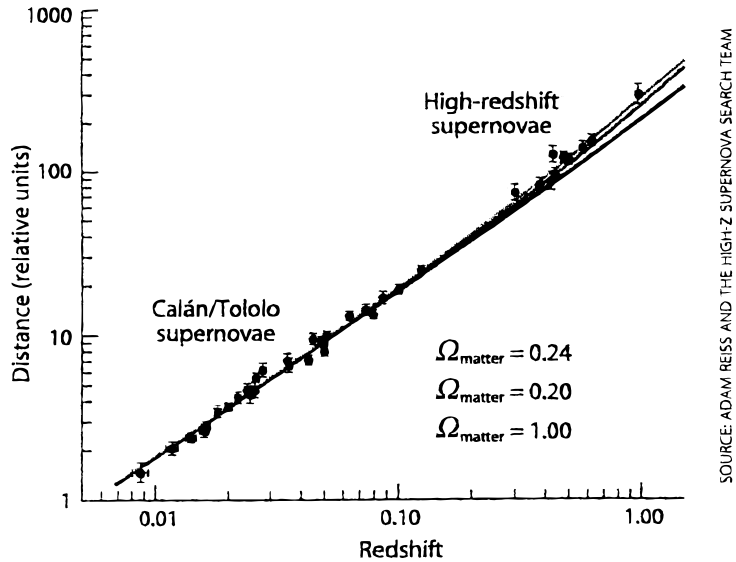

Fig. 10. Distance versus redshift diagram showing the deviation from a constant toward an accelerating expansion. From Riess *et al.* [28].

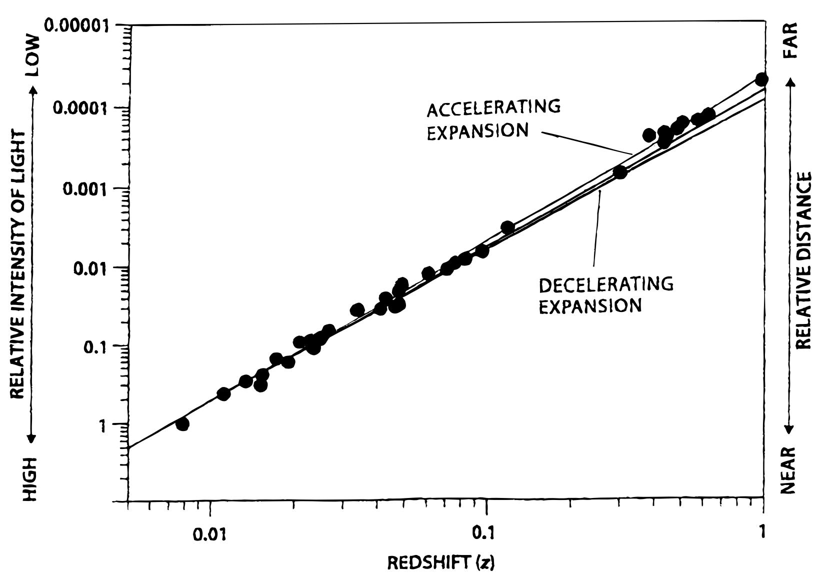

Fig. 11. Relative intensity of light and relative distance versus redshift. From Riess *et al.*, [28].

To summarize, a new general-relativistic theory of cosmology has been presented in which the dynamical variables are those of Hubble, i.e., distances and redshifts. The theory describes the universe as having a three-phase evolution with a decelerating expansion, followed by a constant and an accelerating expansion, and it predicts that the universe is now in the latter phase. As the density of matter decreases, while the universe is at the decelerating phase, it does not have enough time to close up to a big crunch. Rather, it goes to the constant-expansion phase, and then to the accelerating stage.

As we have seen, the equations obtained for the universe expansion are elegant and very simple.

The idea to express cosmological theory in terms of directly measurable quantities, such as distances and redshifts, was partially inspired by Einstein's remarks on the theory of thermodynamics in his Autobiographical Notes [30].

ACKNOWLEDGMENTS

We are grateful to Professor Sir Hermann Bondi for his continuous interest, comments and stimulating suggestions dealing with the contents of this paper. Also, we wish to thank Professor Michael Gedalin for his help in preparing the diagrams.

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